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Bio-inspired Evolutionary Dynamics on Complex Networks under Uncertain Cross-inhibitory Signals [★]

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Abstract

Given a large population of agents, each agent has three possible choices between option 1 or 2 or no option. The two options are equally favorable and the population has to reach consensus on one of the two options quickly and in a distributed way. The more popular an option is, the more likely it is to be chosen by uncommitted agents. Agents committed to one option can be attracted by those committed to the other option through a cross-inhibitory signal. This model originates in the context of honeybee swarms, and we generalize it to duopolistic competition and opinion dynamics. The contributions of this work include i) the formulation of a model to explain the behavioral traits of the honeybees in the case where the interactions are modelled through complex networks, ii) the study of the individual and collective behavior that leads to deadlock or *consensus* depending on a threshold for the cross-inhibitory parameter, iii) the analysis of the impact of the connectivity on *consensus*, and iv) the study of absolute stability for the collective system under time-varying and uncertain cross-inhibitory parameter.

Key words: Decision making; Absolute stability; Agents; Nonlinear systems; Asymptotic stability; Network topologies.

1 Introduction

We consider a large population of agents who can commit to option 1, option 2 or no option (uncommitted state). The two options are equally favorable and the population has to reach consensus on one of the two options quickly and in a distributed way. Agents i) benefit from choosing the more popular option, ii) they can recruit uncommitted agents, and iii) they can send cross-inhibitory signals to agents committed to a different option. As main contribution, we analyze the stability of the individuals' behaviors in a structured environment. The structure of the environment is captured by a complex network, characterized by a given degree distribution. The nodes are the agents and the degree of a node represents its connectivity. Agents are then distributed in classes according to their connectivity, but we consider a general interaction model which make agents in different classes interact between each other. This allows us to understand the impact of heterogeneous con-

nectivity on the collective decision-making process. Our analysis shows that, if the cross-inhibitory parameter exceeds a threshold, which we calculate explicitly, agents reach consensus on one of the two options. Otherwise they distribute uniformly across the two options at the equilibrium. We also extend the model to duopolistic competition and opinion dynamics. The following is a list of additional results with respect to the conference paper, see Stella and Bauso IFAC (2017). First, we provide a convergence analysis as a function of the connectivity. Then, we prove that higher connectivity increases the number of uncommitted agents. Last, we prove absolute stability under time-varying and uncertain cross-inhibitory parameter.

Related literature. The proposed model originates in the context of a swarm of honeybees, see Britton et al. (2002), and Pais et al. (2013). We extend the evolutionary dynamics present in the literature to the structured case, by using complex network theory; by 'consensus' we mean that all agents converge to one option. We frame this study within the context of multi-agent systems as all agents are autonomous, namely they get information on the distribution and choose their strategy accordingly. Differently from Pais et al. (2013), here we stress a different perspective based on the Lyapunov's direct

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method. Evolutionary dynamics in structured environment is discussed in Tan et al. (2014), Ranjbar-Sahraei et al. (2014). Consensus and games are studied in ?. The model developed in this paper shares similarities with the 3-valued logical network in Cheng et al. (2015), although our approach differs in that we use complex networks to model the interaction topology of agents and the model is in continuous-time. We have studied the corresponding mean-field game in Stella and Bauso MED (2017), Stella and Bauso CDC (2017). We have studied the corresponding asymmetric game model in Stella and Bauso (2018). The following articles motivate the interest in the control community for bio-inspired models, see Gray et al. (2018) and Srivastava and Leonard (2017).

This paper is organized as follows. In Section 2, we describe the dynamics under interaction topology. In Section 3, we discuss applications. In Section 4, we analyze consensus and stability. In Section 5, we study absolute stability under nonlinear cross-inhibitory signal. In Section 6, we provide numerical analysis. In Section 7, conclusions are drawn and future directions are discussed.

2 Evolutionary Model

Given a large population of agents, let a complex network be given where $P(k)$ is the probability distribution of the node degrees. Also let x_i^k be the portion of the population with k connections (class k in short) using strategy i , and let $\psi_k = \frac{k}{k_{max}}$ be the parameter capturing the connections of the agents in the network, where k_{max} is the value corresponding to a fully connected network. Furthermore, let $\langle k \rangle$ be the mean value of k , and let $\theta_i := \frac{1}{\langle k \rangle} \sum_k k P(k) x_i^k$ be the probability that a link randomly chosen will point to an agent using strategy i . Therefore, we consider a general interaction model where an agent with k connections interacts with agents of other classes through θ_i . For every class $k \in \mathbb{Z}^+$ the model is:

$$\begin{cases} \dot{x}_1^k = (1 - x_1^k - x_2^k)(\psi_k r \theta_1 + \gamma) - x_1^k(\alpha + \psi_k \sigma \theta_2), \\ \dot{x}_2^k = (1 - x_1^k - x_2^k)(\psi_k r \theta_2 + \gamma) - x_2^k(\alpha + \psi_k \sigma \theta_1), \end{cases} \quad (1)$$

where α , γ and σ are the parameters describing the spontaneous commitment, abandonment and cross-inhibitory signal, respectively. We can view the above system as a microscopic model of the agents in class k parametrized by the macroscopic parameters θ_1 and θ_2 . Such a system, in the asymmetric case, where the parameters are different for the two options, admits the Markov chain representation displayed in Fig. 1.

3 Examples

This section discusses three examples of applications of the model in (1), namely honeybee swarm, duopolistic competition and opinion dynamics.

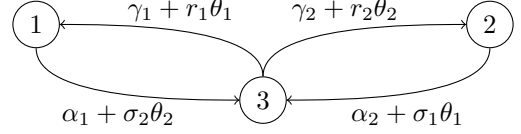


Fig. 1. Markov chain representation of the dynamics in the asymmetric case, i.e. different parameters for each option.

Swarm of honeybees. An unstructured version of system (1) was first developed in the context of honeybee swarms, see Pais et al. (2013). The swarm has to choose between two nest-boxes. The two options have same value $r_1 = r_2 =: r$. Scout bees recruit uncommitted bees via a “waggle dance”. The parameters $\sigma_1 = \sigma_2 =: \sigma$ weigh the strength of the cross-inhibitory signals. We can interpret x_1 as the portion of swarm selecting option 1, x_2 the portion of swarm selecting 2 and x_3 the portion of swarm in the uncommitted state 3. Note that in the unstructured case, the agents are not clustered as they have same connectivity and therefore the index k used in (1) is dropped. Transitions from option 3 to 1 involve a γ_1 amount of independent bees that choose 1 spontaneously and a quantity $\rho_1 x_1$ of bees attracted by those who are already in 1. On the other hand, consider the case where bees move from strategy 1 to 3: α_1 are those that spontaneously abandon their commitment to strategy 1 and $\sigma_2 x_2$ takes into account the cross-inhibitory signal sent from bees using option 2.

Duopolistic competition in marketing. System (1) provides an alternative model of duopolistic competition in marketing, see e.g. Example 9, p. 27 in Bresnan (2010). The classical scenario captured by the well-known *Lanchester* model is as follows. Two manufacturers produce the same product in the same market. The variables x_i represent the market share of the manufacturer i at time t . The cross-inhibitory signal and the “waggle dance” term describe different advertising efforts, which may enter the problem as parameters or controlled inputs in the analysis or design of the advertising campaign. Thus system (1), likewise the *Lanchester* model, describes the evolution of the market share. Because of the interaction topology, system (1) captures the *social influence* of the advertisement campaigns of both manufacturers. A stronger cross-inhibitory signal can be used to model the capability of reaching out to a larger number of potential clients.

Opinion dynamics. Consider a population of individuals, each of which can prefer to vote *left* or *right*, see Hegselmann and Krause (2002). This is represented by the Markov chain depicted in Fig. 1 where nodes 1 and 2 represent the *left* and *right*. The distribution of individuals in each state is subject to transitions from one state to the other. Persuaders who campaign for the *left* can influence the transitions from the *right* to the *uncommitted state*. At the same time *uncommitted* individuals select *left* or *right* proportionally to the level of

popularity of the two options. Because of the interaction topology, system (1) captures the *social influence* of each individual. In other words, the cross-inhibitory signal is stronger for those individuals who have more connections.

4 Consensus and stability

In this section we analyze the stability properties of the model and study the threshold for consensus. Let us consider game dynamics (1) and analyze the mean-field response obtained for a given class of agents assuming that the distribution of the rest of the population is fixed. We can rewrite system (1) in matrix form and, under the assumption that $\theta_1 = \theta_2 =: \theta$, we have

$$\begin{bmatrix} \dot{x}_1^k \\ \dot{x}_2^k \end{bmatrix} = \overbrace{\begin{bmatrix} -(r+\sigma)\psi_k\theta - \alpha - \gamma & -\psi_k r\theta - \gamma \\ -\psi_k r\theta - \gamma & -(r+\sigma)\psi_k\theta - \alpha - \gamma \end{bmatrix}}^{A_k(\theta)} \cdot \begin{bmatrix} x_1^k \\ x_2^k \end{bmatrix} + \underbrace{\begin{bmatrix} \psi_k r\theta + \gamma \\ \psi_k r\theta + \gamma \end{bmatrix}}_{c_k(\theta)}. \quad (2)$$

Theorem 1 *Given an initial state \mathbf{x}_0^k , for all classes k , system (2) is locally asymptotically stable and convergence is faster with increasing connectivity ψ_k . Furthermore, in the cases of no connectivity $\psi_k = 0$ and full connectivity $\psi_k = 1$, system (2) has eigenvalues*

$$\lambda_{1,2} = \begin{cases} (-\alpha - 2\gamma, -\alpha), & \text{for } \psi_k = 0, \\ (-(2r + \sigma)\theta - \alpha - 2\gamma, -\sigma\theta - \alpha), & \text{for } \psi_k = 1. \end{cases}$$

Remark 1 *In light of the above result, note that a higher connectivity: i) speeds up convergence to one of the options in the case of the bees, ii) accelerates convergence of the clients' choices to one of the product, which then become dominant in the market in the case of duopolistic competition in marketing, iii) facilitates a quicker agreement in the case of opinion dynamics. As it can be seen in Fig. 2, the connectivity shifts the eigenvalues further away from the origin (the ones for the case of no connectivity are labelled above the x -axis, while the ones for the case of full connectivity are below).*

Theorem 2 *Given an initial state \mathbf{x}_0^k , for class k , the equilibrium points are $[\hat{x}_1^k, \hat{x}_2^k]^T = -A_k^{-1}(\theta)c_k(\theta)$. Furthermore, at the equilibrium, the distribution of uncommitted agents increases with connectivity ψ_k .*

Remark 2 *By increasing the connectivity of the network, we increase the steady-state percentage of uncommitted bees, undecided clients in duopolistic competition or undecided individuals in opinion dynamics.*

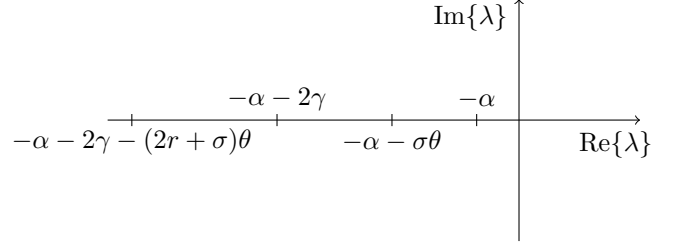


Fig. 2. Change of the eigenvalues for system (2).

Let us now develop a model combining a macroscopic and microscopic dynamics. By averaging on both sides of (2) using $\frac{1}{\langle k \rangle} \sum_k k P(k)$ we have the following macroscopic model:

$$\begin{cases} \dot{\theta}_1 = \frac{r\theta_1}{k_{max}} \left(\frac{V(k)}{\langle k \rangle} - \Psi_1 - \Psi_2 \right) - \frac{\sigma\theta_2}{k_{max}} \Psi_1 \\ \quad -\theta_1\alpha + \gamma - \theta_1\gamma - \theta_2\gamma, \\ \dot{\theta}_2 = \frac{r\theta_2}{k_{max}} \left(\frac{V(k)}{\langle k \rangle} - \Psi_1 - \Psi_2 \right) - \frac{\sigma\theta_1}{k_{max}} \Psi_2 \\ \quad -\theta_2\alpha + \gamma - \theta_1\gamma - \theta_2\gamma, \end{cases} \quad (3)$$

where $V(k) = \sum_k k^2 P(k)x^k$ and $\Psi = \frac{1}{\langle k \rangle} \sum_k k^2 P(k)x^k$.

Theorem 3 *Given an initial state \mathbf{x}_0 , the symmetric equilibrium point in the case of structured environment is locally asymptotically stable if and only if*

$$\sigma < 2r - \frac{rV(k)}{\langle k \rangle \Psi} + \frac{\alpha k_{max}}{\Psi}. \quad (4)$$

The above threshold for the cross-inhibitory signal generalizes the results in the case of unstructured environment, see Stella and Bauso IFAC (2017). When $k = k_{max}$, i.e. in the case of fully connected network, the threshold in (4) coincides with the one found for unstructured environment, see (Stella and Bauso IFAC, 2017, (13)).

5 Uncertain cross-inhibitory coefficient

In this section, we show that stability properties are not compromised even if the cross-inhibitory coefficient σ is uncertain and changes with time within a pre-specified interval. To do this, we first isolate the nonlinearity related to the cross-inhibitory signal in the feedback loop and prove absolute stability using the Kalman-Yakubovich-Popov lemma, see Chapter 10.1 in Khalil (2002). The feedback scheme used in this section is depicted in Fig. 3. We now consider the system where the parameter describing the interactions, i.e. ψ_k can be approximated by 1. Thus, we can rewrite system (1) as:

$$\begin{aligned} \dot{x}_1 &= (1 - x_1 - x_2)(rx_1 + \gamma) - x_2(\sigma x_1 + \alpha), \\ \dot{x}_2 &= (1 - x_1 - x_2)(rx_2 + \gamma) - x_1(\sigma x_2 + \alpha). \end{aligned} \quad (5)$$

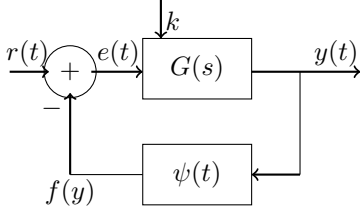


Fig. 3. Feedback scheme used to isolate the nonlinearity.

In the following assumption we introduce the sector nonlinearities.

Assumption 1 Let the cross-inhibitory coefficient σ be in $[0, \tilde{k}]$.

We assume, for simplicity, that $x_1 = x_2$. Thus, we can write

$$\dot{x} = (1 - 2x)(rx + \gamma) - x(\sigma x + \alpha). \quad (6)$$

The linearized version of (5) is

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \underbrace{\begin{bmatrix} r - 3rx - \gamma - \alpha & -rx - \gamma \\ -rx - \gamma & r - 3rx - \gamma - \alpha \end{bmatrix}}_A \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}. \quad (7)$$

Building on the Kalman-Yakubovich-Popov lemma, absolute stability is linked to strictly positive realness of $Z(s) = \mathbb{I} + KG(s)$ where $K = \tilde{k}\mathbf{1}\mathbf{1}^T \in \mathbb{R}^{2 \times 2}$ and $G(s)$ is the transfer function of system (7), yet to be calculated. Before addressing absolute stability, we first investigate conditions under which matrix A is Hurwitz. To be Hurwitz, the trace of matrix A must be negative, i.e.

$$\text{Tr}(A) = r - 3rx - \gamma - \alpha \leq r(1 - 3/2) - \gamma - \alpha,$$

where the equality holds from the condition $x_1 = x_2 =: x$ which implies, in turn, that x can be at most 0.5. In the case where x is sufficiently small, γ and α can be set sufficiently large to guarantee the condition $\text{Tr}(A) < 0$. Furthermore the determinant must be positive, i.e.

$$\Delta(A) = 3rx + \gamma + \alpha - r - rx - \gamma = 2rx + \alpha - r > 0,$$

which is satisfied, when $x = 0.5$, and is still true by choosing a proper α in all the other cases. Now, we isolate the nonlinearities in ψ , and we set $B = C = \mathbb{I}$, where \mathbb{I} denotes the identity matrix. Let us now obtain the transfer function associated with system (7):

$$G(s) = c^T[s\mathbb{I} - A]^{-1}b = \frac{1}{a^2 - b^2} \begin{bmatrix} a & -b \\ -b & a \end{bmatrix}, \quad (8)$$

where $a = s + 3rx + \gamma + \alpha - r$ and $b = rx + \gamma$. Then, for $Z(s)$ we obtain

$$\begin{aligned} Z(s) &= \mathbb{I} + KG(s) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} \frac{ak-bk}{a^2-b^2} & \frac{-bk+ak}{a^2-b^2} \\ \frac{-bk+ak}{a^2-b^2} & \frac{ak-bk}{a^2-b^2} \end{bmatrix} \\ &= \begin{bmatrix} 1 + \frac{k}{a+b} & \frac{k}{a+b} \\ \frac{k}{a+b} & 1 + \frac{k}{a+b} \end{bmatrix} = \frac{1}{s+\zeta} \begin{bmatrix} s+\zeta+k & k \\ k & s+\zeta+k \end{bmatrix}, \end{aligned} \quad (9)$$

where $\zeta = 4rx + 2\gamma + \alpha - r$. We are ready to establish the following result.

Theorem 4 Let system (7) be given and assume that A is Hurwitz. Furthermore, let us consider the sector nonlinearity as in Assumption 1. Then, $Z(s)$ is strictly positive real and the system (7) is absolutely stable.

6 Numerical Simulations

In this section we simulate the system in the case of structured environment, using the Barabási-Albert complex network. We assume that only a few nodes have high connectivity, whereas a large number of nodes have very low connectivity. We use a discretized version of the following power-law distribution, see Moreno et al. (2002):

$$P(k) = \frac{2m^2}{k^3} \quad \text{for } k \geq m, \quad m = \langle k \rangle / 2. \quad (10)$$

In the rest of the section, we write $k_i = N\%$ to mean that agents in class k_i are connected to $N\%$ of the population. The sum of all agents of all classes is in accordance with (10), i.e. $\sum_i k_i = 1$, for all i . The complex network is depicted in Fig. 4.

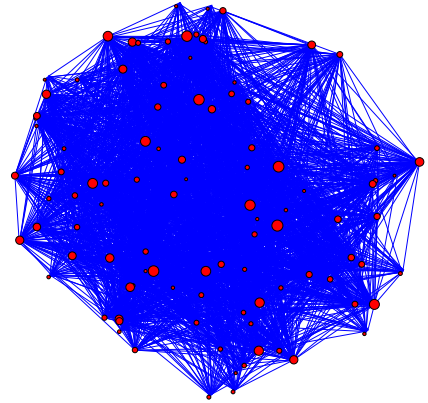


Fig. 4. Complex network used for the simulations.

The following set of simulations involves the study of the evolutionary model in (1), assuming a constant value

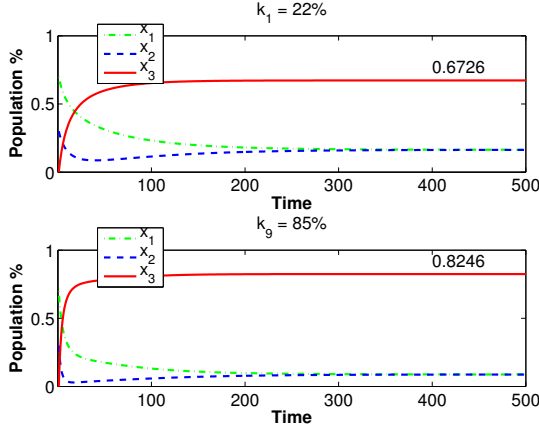


Fig. 5. Time evolution of system when sigma is constant.

$\theta_1 = \theta_2 = 0.4$. We consider two classes of agents, namely those with connectivity $k_1 = 22\%$ and $k_9 = 85\%$, while as for the initial state, the population is split as: 70% in state 1 and 30% in state 2. The plot in Fig. 5 shows that, at the equilibrium, the value of x_3 increases with the connectivity, when σ is constant, namely we have more agents in the uncommitted state. Theorem 2 justifies this behaviour, i.e. the role of parameter ψ_k .

7 Conclusion

For a collective decision making process originating in the context of honeybee swarms, we have provided an evolutionary model and we have studied stability in the case of structured environment, investigating the role of the connectivity in terms of speed of convergence and characterisation of the equilibrium point. Finally, we have analysed the system in case of uncertain cross-inhibitory signal, which generalizes the constant coefficient used in the previous studies. Future works include: i) the study of the connectivity in the case of absolute stability, ii) the analysis of a related model to describe virus propagation in power grids, and iii) the extension of the results to stochastic dynamics with births and deaths in the population.

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Appendix

Proof of Theorem 1. The determinant Δ of matrix $A_k(\theta)$ is always positive. To see this, note that $(r + \sigma)\psi_k\theta + \alpha + \gamma \geq \psi_k r\theta + \gamma$. Also, the trace of the above matrix is negative, and therefore the system is asymptotically stable. From $T^2 - 4\Delta = 4(\psi_k r\theta + \gamma)^2 > 0$, the equilibrium point is asymptotically stable.

As for the speed of convergence, we calculate the determinant, which is given by $T^2 - 4\Delta = 4[\psi_k r\theta + \gamma]^2$. Thus, the eigenvalues of the Jacobian matrix are $\lambda_{1,2} = -(\sigma + r)\psi_k\theta - \alpha - \gamma \pm (\psi_k r\theta + \gamma)$. In the two extreme case of no connectivity $\psi_k = 0$ and full connectivity $\psi_k = 1$:

$$\lambda_{1,2} = \begin{cases} (-\alpha - 2\gamma, -\alpha), & \text{for } \psi_k = 0, \\ (-(2r + \sigma)\theta - \alpha - 2\gamma, -\sigma\theta - \alpha), & \text{for } \psi_k = 1. \end{cases}$$

Proof of Theorem 2. We can compute the following

$$\begin{aligned} x_k^* &= A_k^{-1}(\theta) c_k(\theta) \\ &= \frac{1}{-(2r + \sigma)\psi_k\theta - \alpha - 2\gamma} [-\psi_k r\theta - \gamma \quad -\psi_k r\theta - \gamma]^T \\ &= \frac{1}{(2r + \sigma)\psi_k\theta + \alpha + 2\gamma} [\psi_k r\theta + \gamma \quad \psi_k r\theta + \gamma]^T. \end{aligned} \quad (11)$$

Again, when considering the above two cases we get

$$\begin{aligned} x_k^* &= \frac{1}{\alpha + 2\gamma} [\gamma \quad \gamma]^T \quad \psi_k = 0, \\ x_k^* &= \frac{1}{(2r + \sigma)\theta + \alpha + 2\gamma} [r\theta + \gamma \quad r\theta + \gamma]^T \quad \psi_k = 1. \end{aligned} \quad (12)$$

Therefore, we can also say that higher connectivity increases the number of agents in the uncommitted state.

Proof of Theorem 3. To compute the equilibrium, let us set $\hat{\theta}_1 = \hat{\theta}_2$ and obtain:

$$(\theta_1 - \theta_2) \left(\frac{r}{k_{max}} \Psi_3 - \alpha \right) + \frac{\sigma\theta_1}{k_{max}} \Psi_2 - \frac{\sigma\theta_2}{k_{max}} \Psi_1 = 0. \quad (13)$$

Note that in a symmetric equilibrium where $\theta_1 = \theta_2$, we can neglect the last two terms. We can then compute the Jacobian of system (3). Saddle-points are obtained when the determinant of the Jacobian is less than 0. Then, we take $\Psi_1 = \Psi_2$ and impose that the right-hand side is greater than the left-hand side

$$\left(-\frac{\sigma}{k_{max}} \Psi - \gamma \right)^2 > \left(\frac{r}{k_{max}} (V(k)/\langle k \rangle - 2\Psi) - \alpha - \gamma \right)^2.$$

By taking the square root on both sides, since the left-hand side is strictly negative, we have $-\frac{\sigma}{k_{max}} \Psi < -2\frac{r}{k_{max}} \Psi + \frac{rV(k)}{k_{max}\langle k \rangle} - \alpha$, and after some basic algebra, we get (4).

Proof of theorem 4. Let us prove that $Z(s)$ is strictly positive real. The following conditions must hold true:

- $Z(s)$ is Hurwitz, i.e poles of all elements of $Z(s)$ have negative real parts;
- $Z(j\omega) + Z(-j\omega) > 0, \quad \forall \omega \in \mathbb{R};$
- $Z(\infty) + Z^T(\infty) > 0.$

First, we prove that $Z(s)$ is Hurwitz. Thus, all the poles must be negative, i.e. $r - 4rx - 2\gamma - \alpha < 0$. which holds true, after considering the discussion on the trace of matrix A as a direct consequence. Now, we check the second condition. It follows that

$$\begin{aligned} Z(j\omega) + Z(-j\omega) &= \begin{bmatrix} \frac{j\omega + \zeta + k}{j\omega + \zeta} & \frac{k}{j\omega + \zeta} \\ \frac{k}{j\omega + \zeta} & \frac{j\omega + \zeta + k}{j\omega + \zeta} \end{bmatrix} + \begin{bmatrix} \frac{-j\omega + \zeta + k}{-j\omega + \zeta} & \frac{k}{-j\omega + \zeta} \\ \frac{k}{-j\omega + \zeta} & \frac{-j\omega + \zeta + k}{-j\omega + \zeta} \end{bmatrix} = \begin{bmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{bmatrix}, \end{aligned}$$

where

$$z_{11} = z_{22} = \frac{\omega^2 - j\omega\zeta - j\omega k + j\omega\zeta + \zeta^2 + \zeta k + \omega^2 + j\omega\zeta + j\omega k - j\omega\zeta + \zeta^2 + \zeta k}{\zeta^2 + \omega^2}$$

and $z_{12} = z_{21} = \frac{-j\omega k + \zeta + k + j\omega k + \zeta k}{\zeta^2 + \omega^2}$. Thus, the second condition can be rewritten as

$$Z(j\omega) + Z(-j\omega) = \begin{bmatrix} \frac{2\omega^2 + 2\zeta^2 + 2\zeta k}{\zeta^2 + \omega^2} & \frac{2\zeta + k}{\zeta^2 + \omega^2} \\ \frac{2\zeta + k}{\zeta^2 + \omega^2} & \frac{2\omega^2 + 2\zeta^2 + 2\zeta k}{\zeta^2 + \omega^2} \end{bmatrix} > 0,$$

which is verified for all ω . Last, as $Z(s)$ is symmetric the third condition implies that $2Z(\infty) > 0$. In the limit it converges to an identity matrix, and thus the third condition is verified. Let us turn to prove absolute stability by showing that there exists a Lyapunov function $V(x) = x^T P x$. Let us derive the expression of $\dot{V}(t, x)$ as

$$\begin{aligned} \dot{V}(t, x) &= \dot{x}^T P x + x^T P \dot{x} \\ &= x^T A^T P x + x^T P A x - \psi^T B^T P x - x^T P B \psi, \end{aligned} \quad (14)$$

where ψ is equivalent of writing $\psi(t, y)$. For the condition on the sector nonlinearity $-2\psi^T(\psi - Ky) \geq 0$ and from matrices P and K being symmetric, we can now specialize it to our case, i.e. A symmetric and $B = C = \mathbb{I}$, as

$$\begin{aligned} \dot{V}(t, x) &\leq x^T (A^T P + P A) x - 2x^T P B \psi - 2\psi^T (\psi - Ky) \\ &= 2x^T A P x + 2x^T (K - P) \psi - 2\psi^T \psi. \end{aligned} \quad (15)$$

To show that the right-hand side of (15) is negative, we can construct a square term by imposing

$$2AP = -L^T L - \epsilon P, \quad K - P = \sqrt{2} L^T, \quad (16)$$

where $\epsilon > 0$ is a constant and matrix $P = P^T > 0$. Now, we can rewrite (15) as

$$\begin{aligned} \dot{V}(t, x) &\leq -\epsilon x^T P x - x^T L^T L x + 2\sqrt{2} x^T L^T \psi - 2\psi^T \psi \\ &= -\epsilon x^T P x - [Lx - \sqrt{2}\psi]^T [Lx - \sqrt{2}\psi] \\ &\leq -\epsilon x^T P x. \end{aligned} \quad (17)$$

From Kalman-Yakubovich-Popov lemma, we can obtain P, L, ϵ solving (16), as $Z(s)$ is positive real. This concludes our proof.